JOURNAL OF ALGEBRA **176**, 346–355 (1995)

Right Chain Domains with Prescribed Value Holoid

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Communicated by Kent R. Fuller

Received December 31, 1993

Let $H = H_{i \in I}(G_i, N_i)$ be the split holoid of a family of ordered groups G_i with convex subsets N_i that contain G_i^+, the positive cone of G_i, where I is an ordered index set. Then there exists a right invariant right chain domain R with $H \cong H(R)$, the associated value semigroup of nonzero principal right ideals of R. © 1995 Academic Press, Inc.

Given any ordered group G, there exists a division ring D and a subring V of D with x or x^{-1} in V for every $x \in D$ such that all one-sided ideals of V are two-sided, i.e., V is an invariant chain domain, and such that G is isomorphic to the group of principal fractional V-ideals. This was observed by Krull [K32] using localizations of group algebras over G in the commutative case and by Malcev-Neumann [N49] using generalized power series rings in the general case.

A *right chain domain* R is a ring with identity and without zero-divisors in which for any $a, b \in R$ either $aR \subseteq bR$ or $bR \subset aR$ holds. The nonzero principal right ideals of R form a semigroup $H(R)$ under multiplication if and only if R is right invariant, i.e., $Rr \subseteq rR$ for all $r \in R$.

The semigroup $H(R) = H$ in that case is an ordered semigroup with identity e such that $a < b$ in H holds if and only if $ac = b$ for some $e \neq c \in H$. For reasons of brevity we will call semigroups of this type

*Supported in part by NSERC.

0021-8693/95 $12.00

holoids in this paper, even though we used the term right invariant right holoid in [BT89]. Related structures were investigated among others by Klein-Barmen [KB48], who introduced the term "linear holoid," by Conrad [Con 60], and by Schein [S79].

One can ask whether for a given holoid H there exists a right invariant right chain domain R with $H(R) \cong H$. This question has been answered positively in the case where H has rank 1 or rank 2 (the rank of H is defined as the order type of the totally ordered set of proper convex subsemigroups of H), or if H is right noetherian, in which case $H \cong O_I = \{\alpha \mid \alpha < \omega^I\}$, the semigroup of ordinals less than a power of ω, where ω denotes the order type of the natural numbers with addition as operation. Rings with $H(R) \cong O_I$ for arbitrary well-ordered I were first constructed by Jategaonkar in [J69]; another construction was given by Cohn in [C85].

Jategaonkar's intention in constructing these rings was to provide answers to various open problems in ring theory, in particular to show that there does not exist an ordinal α with $J(R)^\alpha = (0)$ for all right noetherian rings R (for earlier results on Jacobson's conjecture, see [H65]).

In Section 1 the definition of a *split holoid* is given for a family $(G_i, N_i)_{i \in I}$, where the G_i's are ordered groups. Not only can the semigroup O_I be considered as the split holoid for a family consisting of copies of $(\mathbb{Z}, \mathbb{N}_0)$, but in [BT89] we discussed conditions under which a holoid is a split holoid.

The main result of the present paper shows that for a split holoid H there exists a right invariant right chain domain R with $H \cong H(R)$; see Theorem 4. We will use generalized power series rings over suitably chosen subsemigroups of the wreath product of ordered groups. Both the constructions given by Jategaonkar [J69] and Cohn [C85], however, rely on the localization of certain skew semigroup rings. This approach is not available if one deals with the positive cones of ordered groups in general.

In this paper all rings have an identity and are associative, but are not necessarily commutative. We call a ring R a *domain* if R has no zero-divisors. The *Jacobson radical* of the ring R is denoted by $J(R) = J$ whereas $U(R) = U$ is the *group of units* of R. We denote with e_G the identity of a group G.

1. HOLOIDS AND RESTRICTED WREATH PRODUCTS OF ORDERED GROUPS

A semigroup H with identity $e = e_H$ and a (total) order relation $\leq$ is called an *ordered semigroup* if $a \leq b$ implies $ac \leq bc$, $ca \leq cb$ for elements $a, b, c \in H$. If, in addition, $a < b$ holds if and only if $b = ac$ for

$e \neq c$, then H is called a *holoid*. It follows from this definition that $a \geq e$ for all $a \in H$, that for $a, b \in H$ there exists $b' \in H$ with $ba = ab'$, and that $ab = ac$ implies $b = c$. The other cancellation law holds if and only if H is embeddable into an ordered group. We now describe a particular class of holoids which we encountered in [BT89].

Let $G_1, \ldots, G_n$ be ordered groups with $e_i = e_{G_i}$ the identity of G_i, $G_i^+ = \{g_i \in G_i \mid g_i \geq e_i\}$ the positive cone of G_i, and $G_i^+ \subseteq N_i$ a convex subset of G_i for $i \in I = \{1, \ldots, n\}$. We then consider the set $H = H_{i=1}^n(G_i, N_i)$ of n-tuples $h = (g_n, \ldots, g_1)$, $g_i \in N_i$ for $i \in I$ for which either $h = (e_n, \ldots, e_1) = e_H$, or for which the leading component is positive, i.e., $g_m > e_m$, but $g_j = e_j$ for $j > m$. The operation on H is given as follows: Let $h = (g_n, \ldots, g_1)$ and $h' = (e_n, \ldots, e_{k+1}, g'_k, \ldots, g'_1)$ be elements in H with $g'_k > e_k$. Then

$$hh' = (g_n, \ldots, g_{k+1}, g_k g'_k, g'_{k-1}, \ldots, g'_1).$$

The ordering on H is lexicographical, i.e., $h = (e_n, \ldots, e_{j+1}, g_j, \ldots, g_1) > h' = (e_n, \ldots, e_{k+1}, g'_k, \ldots, g'_1)$ for $g_j > e_j$ and $g'_k > e_k$, if and only if either $j > k$ or $j = k$ and $g_j > g'_j$. It then follows that $H = H_{i=1}^n(G_i, N_i)$ is a holoid which we call the split holoid of the family $(G_i, N_i)_{i \in I}$. We assume that each G_i is nontrivial. The holoid H is not embeddable into a group if I contains at least two elements.

The above construction can be extended to a set of groups $\{G_i \mid i \in I\}$ for an ordered index set I, where we assume that only finitely many components of an element $h \in H$ are distinct from the identity. The semigroup O_I of all ordinal numbers less than the power ω^I of ω, with addition as operation, falls into the above described class of split holoids, where $G_i \cong (\mathbb{Z}, +)$ with $N_i = G_i^+ \cong \mathbb{N}_0$, and a well-ordered index set I.

Let A and B be two ordered groups. We define the base group C of the *wreath product* $A \wr B$ as the direct sum of B-indexed copies A_b of A ordered lexicographically. We denote with c_b the b-component of an element $c \in C$. Then $A \wr B$ is defined as the semidirect product of B with C, i.e.,

$$W = A \wr B = \{b'c \mid b' \in B, c \in C \text{ and } cb' = b'c^{b'}, (c^{b'})_b = c_{bb'^{-1}}\}.$$

The group W is again an ordered group with the lexicographical ordering and $b'c > e_W$ if $b' > e_B$ or $b' = e_B$ and $c_b > e_A$ for b minimal with $c_b \neq e_A$ (see, for example, [F67]). The group B is naturally embedded into W by identifying be_C in W with $b \in B$, and similarly A is embedded into W if the element $e_B c$, with $c_{e_B} = a \in A$ and c_b equal e_A otherwise, is identified with $a \in A$. Of course, the order of W extends the order of A as well as the order of B and $b > a$ in W for every $b > e_B$ and $a \in A$.

Given ordered groups $G_1, \ldots, G_n$ with convex subsets $G_i^+ \subseteq N_i \subseteq G_i$ for $i \in I = \{1, \ldots, n\}$, we consider the wreath product $W = W_n = (\ldots((G_1 \wr G_2) \wr G_3) \wr \cdots \wr G_n)$. Next we define a subgroup $S = S_n$ of $W = W_n$ by induction as follows. Let $S_1 = \{e_1\}$ and $S_k = \{x = e_k c \in W_k \mid c_g = e_{W_{k-1}}$ for $g < e_k \in G_k$ and $c_{e_k} = s_{k-1} \in S_{k-1}\}$. Every element $x \in S_k$ can be written as $x = e_k c = e_k s_{k-1} x_k$ where $(x_k)_g = e_{W_k}$ for $g \leq e_k \in G_k$. The set of all such elements x_k is denoted by X_k. Each X_k is a subgroup of W_n. It follows that every element $s \in S_n = S$ has the form $s = e_n \ldots e_1 x_1 \ldots x_n = x_1 \ldots x_n$ with $e_k \in G_k \subseteq W$ and uniquely determined $x_k \in X_k \subseteq W$, $k \in I$. Hence, $S = X_1 \ldots X_n$ with $X_1 = \{e_1\}$.

LEMMA 1. *The following rules hold for elements $x_k \in X_k$ and $g_i \in G_i$:*

(i) *For all $g_j \in G_j^+ \backslash \{e_j\}$, $g_i \in G_i$, and $i < j \leq n$ we have $g_j^{-1} g_i g_j \in X_j$.*

(ii) *For all $x_i \in X_i$, $x_j \in X_j$, $i \neq j$ we have $x_i x_j = x_j x_i$.*

(iii) *For all $g_i \in G_i$, $x_j \in X_j$, $i < j$ we have $g_i^{-1} x_j g_i = x_j \in X_j$.*

(iv) *For all $g_j \in G_j^+ \backslash \{e_j\}$, $x_i \in X_i$, $i \leq j$ we have $g_j^{-1} x_i g_j \in X_j$.*

Proof. We consider $W_j = W_{j-1} \wr G_j$ with the base group C and observe that the elements $g_i \in G_i$ and $x_i \in X_i$ occur in the e_j-component of C for $i < j$.

(i) The element $g_j^{-1} g_i g_j$ is contained in the g_j-component of C and hence in X_j for $g_j > e_j$.

(ii) and (iii) follow since x_j has a trivial e_j-component in C.

(iv) The element $g_j^{-1} x_i g_j$ has trivial g-components in its decomposition in C for all $g \in G_j$ with $g < g_j$ and is therefore contained in X_j for $g_j > e_j$. ∎

We consider the earlier constructed split holoid $H = H_{i=1}^n(G_i, N_i)$. The embeddings of the G_i's into W afford also an embedding of the underlying ordered set of the holoid H into W. We denote the image of $h \in H$ by $\hat{h} \in W$ and observe that $h \to \hat{h}$ is an order-preserving embedding, but not a semigroup monomorphism for $n > 1$. However, the next result shows that for $h, h' \in H$ the elements $\hat{h}\widehat{h'}$ and $\widehat{hh'}$ differ in W only by some $s \in S$.

LEMMA 2. *Let H be the holoid and $S \subset W$ be the groups defined above. We obtain:*

(i) *If $\hat{h}s = \widehat{h'}s'$ for $s, s' \in S$, $h, h' \in H$, then $h = h'$ and $s = s'$.*

(ii) *For all $e_H \neq h \in H$, $s \in S$ we have $\hat{h} > s \in W$.*

(iii) *For all $h \in H$ we have $\hat{h}^{-1} S \hat{h} \subseteq S$ and $g_j^{-1} g_i g_j \in S$ for $e_j \neq g_j \in G_j^+$, $g_i \in G_i$ and $j > i$.*

(iv) *If $h, h' \in H$, $s, s' \in S$, then there exists $s'' \in S$ with $\hat{h} s \widehat{h'} s' = \widehat{hh'} s''$ in W.*

(v) *$\mathscr{H} = \{\hat{h}s \mid h \in H,\ s \in S\}$ is a subsemigroup of W with $\hat{h}^{-1}\mathscr{H}\hat{h} \subseteq \mathscr{H}$ for any $h \in H$.*

Proof. (i) and (ii) are immediate consequences of the definitions of $\hat{H}$, S, and the order in W.

(iii) We prove by induction on k that $\hat{h}^{-1}s\hat{h} \in S$ for $\hat{h}, s \in W_k$. This is true for $k = 1$. For $\hat{h} = g_k \dots g_1$ with $e_k \neq g_k \in G_k^+$, $g_i \in N_i$, $i = 1, \dots, k$, it follows that

$$\hat{h}^{-1}s\hat{h} = g_1^{-1} \dots g_{k-1}^{-1}(g_k^{-1}sg_k)g_{k-1} \dots g_1 = g_1^{-1} \dots g_{k-1}^{-1}y_k g_{k-1} \dots g_1$$

with $g_k^{-1}sg_k = y_k \in X_k$ (Lemma 1(iv)) since $S = X_1 \dots X_k$. The element $y_k \in G_k$ commutes with g_j for every $j < k$ by Lemma 1(iii) and $\hat{h}^{-1}s\hat{h} = g_1^{-1} \dots g_{k-1}^{-1}y_k g_{k-1} \dots g_1 = y_k \in S$ follows.

If $\hat{h} = g_r \dots g_1$ with $e_r \neq g_r \in G_r^+$ and $r < k$ and $s = x_1 \dots x_k$, then

$$\begin{aligned}\hat{h}^{-1}s\hat{h} &= (g_r \dots g_1)^{-1}(x_1 \dots x_k)(g_r \dots g_1)\\ &= \left[(g_r \dots g_1)^{-1}(x_1 \dots x_{k-1})(g_r \dots g_1)\right]\left[(g_r \dots g_1)^{-1}x_k(g_r \dots g_1)\right].\end{aligned}$$

The element in the first square bracket is in S by induction, and $(g_r \dots g_1)^{-1}x_k(g_r \dots g_1) = x_k$, since $g_i^{-1}x_k g_i = x_k$ for $i < k$ by Lemma 1(iii). The second statement follows from Lemma 1(ii).

(iv) We have $\hat{h}s\widehat{h'}s' = \hat{h}\widehat{h'}\widehat{h'}^{-1}s\widehat{h'}s' = \hat{h}\widehat{h'}s''s'$ for $s'' = \widehat{h'}^{-1}s\widehat{h'} \in S$ by (iii). It remains to show that $\hat{h}\widehat{h'} = \widehat{hh'}t$ for $t \in S$. Let $\hat{h} = g_n \dots g_1$ and $\hat{h}' = g'_k \dots g'_1$ with $e_k \neq g'_k \in G_k^+$. Then $\hat{h}\widehat{h'} = (g_n \dots g_1)(g'_k \dots g'_1)$. Since $g_j^{-1}g_i g_j \in S$ for $e_j \neq g_j \in G_j^+$, $g_i \in G_i$, $j > i$ we have

$$\hat{h}\widehat{h'} = g_n \dots g_k g'_k(g_k'^{-1}g_{k-1}g'_k)(\dots)(g_k'^{-1}g_1 g'_k)g'_{k-1} \dots g'_1,$$

where the elements $g_k'^{-1}g_i g'_k$ are in X_k for $i < k$ by Lemma 1(i) and therefore commute with every element $g'_j \in G_j$ for $j < k$ (see Lemma 1(iii)). This proves (iv).

(v) By (iv), $\mathscr{H}$ is a subsemigroup of W. We consider $\hat{h}^{-1}\widehat{h'}s\hat{h}$ for $\widehat{h'}s \in \mathscr{H}$, $h \in H$. Since

$$\hat{h}^{-1}\widehat{h'}s\hat{h} = \hat{h}^{-1}\widehat{h'}\hat{h}\hat{h}^{-1}s\hat{h}$$

and $\hat{h}^{-1}s\hat{h} \in S$ by (iv), the statement follows if we show $\widehat{h'}\hat{h} = \hat{h}\widehat{h''}s''$ for some $h'' \in H$, $s'' \in S$. We have $h'h = hh''$ in the holoid H for some element $h'' \in H$. We obtain, by applying (iv) twice, the following equations: $\widehat{h'}\hat{h} = \widehat{h'h}t = \widehat{hh''}t = \hat{h}\widehat{h''}t't$ with $t, t' \in S$. ∎

Let I be an ordered index set and let $\mathcal{F}$ be the family of finite subsets of I. For the set $\{G_i \mid i \in I\}$ of ordered groups G_i with convex subsets $G_i \supseteq N_i \supseteq G_i^+$ we define for every element $\alpha = \{i_1, \dots, i_n\} \in \mathcal{F}$ with $i_1 < i_2 < \cdots < i_n$ the split holoid $H_\alpha = H_{i_t \in \alpha}(G_{i_t}, N_{i_t})$ and the wreath product $W_\alpha = (\dots((G_{i_1} \wr G_{i_2}) \wr \cdots \wr G_{i_n}))$. By Lemma 2 there exists a subsemigroup $\mathcal{H}_\alpha$ containing a subgroup S_α, and an order-preserving embedding $\hat{}_\alpha$ from $(H_\alpha, \leq)$ into $(\mathcal{H}_\alpha, \leq)$ such that properties (i)–(v) in Lemma 2 hold.

If α is a subset of $\beta = \{j_1, \dots, j_r\}$, for $\alpha, \beta \in \mathcal{F}$ with $j_1 < j_2 < \cdots < j_r$, then there exists an embedding of H_α into H_β and, as we show below, an embedding of W_α into W_β which induces embeddings of S_α into S_β and of $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$, respectively; the restriction of $\hat{}_\alpha$ to H_α is equal to $\hat{}_\beta$. Clearly, H_α can be considered as a subholoid of H_β and we use induction on r and n as well as the embeddings of the groups A and B into $A \wr B$ to define the embedding of W_α into W_β. If $\alpha \subseteq \beta' = \beta \setminus \{j_r\}$ then an embedding exists from W_α into $W_{\beta'}$ by induction, and of $W_{\beta'}$ into $W_\beta = W_{\beta'} \wr G_{j_r}$. If $i_n = j_r$, then $W_\alpha = W_{\alpha'} \wr G_{j_r}$ for $\alpha' = \alpha \setminus \{j_r\}$ and $W_{\alpha'}$ can be embedded into $W_{\beta'}$ by induction and hence W_α can be embedded into W_β. The direct limit $\lim_{\to \alpha \in \mathcal{F}} H_\alpha$ exists and is the split holoid $H = H_{i \in I}(G_i, N_i)$ of the family $(G_i, N_i)_{i \in I}$. Similarly, the direct limit $\lim_{\to \alpha \in \mathcal{F}} W_\alpha$ exists and is an ordered group; we denote it with W, the wreath product of the groups $\{G_i \mid i \in I\}$ and W contains the subsemigroup $\mathcal{H} = \lim_{\to \alpha \in \mathcal{F}} \mathcal{H}_\alpha$ and $S = \lim_{\to \alpha \in \mathcal{F}} S_\alpha \leq \mathcal{H}$.

COROLLARY 3. *Let $H = H_{i \in I}(G_i, N_i)$ be the split holoid of a family of ordered groups G_i with convex subsets $G_i^+ \subseteq N_i$ where I is an ordered index set. Then there exist a subsemigroup $\mathcal{H}$ of the wreath product W of the groups G_i, a subgroup S of $\mathcal{H}$, and an order-preserving embedding $\hat{}$ from $(H, \leq)$ into $\mathcal{H}$ such that properties* (i)–(v) *in Lemma 2 hold.*

Proof. Statements (i)–(v) hold for each H_α, W_α, $\mathcal{H}_\alpha$, S_α, and $\hat{}_\alpha$ for all $\alpha \in \mathcal{F}$, and hence for H, W, $\mathcal{H}$, S, and $\hat{}$. ∎

2. THE CONSTRUCTION OF THE RINGS

We can now prove the main result.

THEOREM 4. *Let G_i, $i \in I$, be a set of ordered groups G_i for a (totally) ordered index set I and let N_i be a convex subset of G_i which contains the positive cone G_i^+ of G_i for each i. Then there exists a right invariant right chain domain R such that $H(R)$, the value semigroup of R, is isomorphic to* *the split holoid $H = H_{i \in I}(G_i, N_i)$. The ring R is obtained as a suitable subring of the generalized power series ring over the wreath product of the G_i.*

Proof. Given the set $(G_i, i \in I)$ of ordered groups we can construct the wreath product W of the G_i's with the subsemigroup $\mathscr{H}$ determined by H, the subgroup $S \subseteq \mathscr{H}$, and the mapping $\hat{}$ from H into $\mathscr{H}$ such that (i)–(v) in Lemma 2 hold. Since W is an ordered group, there exists $E = \mathbb{Q}\{\{W\}\}$, the ring of generalized power series $\Sigma w q_w$, where $q_w \in \mathbb{Q}$, the field of rational numbers, with well-ordered support $\{w \mid q_w \neq 0\}$ (see [N49]). Since $\mathscr{H}$ is a subsemigroup of W (Lemma 2(v)), it follows that $R = \{\Sigma w q_w \in E \mid w \in \mathscr{H}\}$ is a subring of E.

Let $a = \Sigma w q_w$ be an element in R and let $w_0 = \min\{w \mid q_w \neq 0\}$ with $w_0 = \widehat{h_0} s_0$ for $h_0 \in H$, $s_0 \in S$. We show that $a = \widehat{h_0} u$ for a unit $u \in R$. We prove by contradiction that $\widehat{h_0} \leq \hat{h}$ for $w = \hat{h}s$ in the support of a. From the assumption $\hat{h} < \widehat{h_0}$ it follows that there exists an element $e_H < h' \in H$ with $hh' = h_0$. Thus $\widehat{hh'} = \widehat{h_0}$, and by Lemma 2(iv) $\hat{h}\widehat{h'}s' = \widehat{h_0}$ for some $s' \in S$. This is a contradiction, since $\widehat{h'} > s s_0^{-1} s'^{-1}$ (use Lemma 2(ii)) which implies $\widehat{h_0} = \hat{h}\widehat{h'}s' > \hat{h} s s_0^{-1} > \widehat{h_0}$, since $w = \hat{h}s > \widehat{h_0} s_0$.

For $w_0 < w \in \mathscr{H}$ there exists an element $w' \in W$ with $w_0 w' = \widehat{h_0} s_0 w' = w = \hat{h}s$. Since $\widehat{h_0} \leq \hat{h}$ in W, we have $h_0 \leq h$ in H and so $h_0 h' = h$ for some h' in the holoid H. Hence, $\widehat{h_0} s_0 w' = \widehat{h_0 h'} s = \widehat{h_0}\widehat{h'} t s$ for some $t \in S$ (Lemma 2(iv)). We obtain $w' = s_0^{-1} \widehat{h'} t s \in \mathscr{H}$ by Lemma 2(v) and

$$a = \sum w q_w = \widehat{h_0} s_0 q_{w_0}\Bigl(1 + \sum_{w' > e_W} w' q'_{w'}\Bigr)$$

follows. The element $1 + \Sigma w' q'_{w'} = 1 - m$ is in R and has the inverse $1 + \Sigma m^i$ in E which is also an element of R since its support is in $\mathscr{H}$; we conclude that $a = \widehat{h_0} u$ for a unit u in R, since s_0 has an inverse in R and $0 \neq q_{w_0} \in \mathbb{Q}$.

On the other hand, no element $\hat{h}$ with $h \neq e$ is a unit in R (use Lemma 2(i)). Hence every principal right ideal in R is of the form $\hat{h}R$ and R is a right chain domain with $\hat{h}R \subset \widehat{h'}R$ if and only if $h > h'$ in H. It follows from Lemma 2(iii) and (v) that R is right invariant which implies $\hat{h}R\widehat{h'}R = \hat{h}\widehat{h'}R = \widehat{hh'}sR = \widehat{hh'}R$, using Lemma 2(iv), i.e., $H(R) \cong H$ as ordered semigroups. ∎

We note that the field $\mathbb{Q}$ of rational numbers in the above construction can be replaced by any skew field.

The rings R constructed in the last theorem are *local right Bezout domains*. If I contains more than one element, then the ring R is not a left Ore domain, since $Rg_2 \cap Rg_2 g_1 = (0)$ for $g_i \in G_i^+ \setminus \{e_i\}$, $i = 1, 2$, and hence

R is not a left Bezout domain (see also [Sm67]). However, every finitely generated left ideal is a free R-module of unique rank, since R is a semifir (see [C85, p. 65]). The left projective dimension of R can, however, be arbitrarily large; see [B69].

If the index set I contains exactly one element, say $I = \{1\}$, then the ring R in the above theorem is the *chain domain* with value group G_1 as constructed by Neumann.

If the index set I is well ordered and $(G_i, N_i) = (\mathbb{Z}, \mathbb{N}_0)$ for all $i \in I$, then the ring R in the theorem is a *right principal ideal ring*. We obtain a representation of the holoid O_I as the semigroup $H(R)$ of a right invariant right chain domain, a problem that was solved by Jategaonkar [J69]. Conversely, if R is a right noetherian right invariant right chain domain, then $H(R) \cong O_I$ for some well-ordered index set I (see [B69]) and O_I is isomorphic to the split holoid $H = H_{i \in I}(\mathbb{Z}, \mathbb{N}_0)_i$. It follows (see [J69]) that the rings constructed here for the particular split holoids $O_I \cong H = H_{i \in I}(\mathbb{Z}, \mathbb{N}_0)$, with I well ordered, can serve as counterexamples for the Jacobson conjecture, to illustrate various phenomena connected with unique factorization and primary right principal right ideals, and to provide examples of rings R with right projective dimension $R = 1$ and left projective dimension $R \geq n$ for any given n.

Since the prime ideals P in the right noetherian case are principal, we have $P^2 \neq P$ for $P \neq (0)$. Arbitrary right chain domains for which this condition holds are called *discrete*. The set of prime ideals in a discrete right chain domain is inversely well ordered by inclusion. This follows since all prime ideals are completely prime in this case, the union of prime ideals P_ρ is again a prime P, and $P = P_{\rho_0}$ for some ρ_0, since $P \neq P^2$.

Next we describe the class of split holoids $H = H_{i \in I}(G_i, N_i)$ which occur as the holoids $H(R)$ of discrete right invariant right chain domains. For each $i \in I$ the split holoid $H_i = H_{j < i}(G_j, N_j)$ which can be considered as a convex subsemigroup of H with corresponding prime ideal P_i of R equal to $(\bar{h} \mid h \in H \setminus H_i)$ where $\bar{h}$ is the principal right ideal of R that corresponds to $h \in H$ for the canonical isomorphism between H and $H(R)$. Since we have $H_i < H_j$ for $i < j$, but $P_i \supset P_j$, it follows that I is well ordered if R is discrete.

Similarly, it follows that for each G_i the chain of convex subgroups (see [F67, p. 79]) must be well ordered by inclusion with $C'/C \cong \mathbb{Z}$ for every subgroup C of G_i where C' is the minimal convex subgroup of G_i containing C.

Conversely, if the above conditions concerning I and G_i are satisfied, then $H(R) = H_{i \in I}(G_i, N_i)$ is discrete for arbitrary convex subsets $G_i^+ \subseteq N_i$ of G_i.

We conclude with an example that illustrates the influence of the denominator sets N_i and shows that a right invariant right chain domain can contain several prime ideals that are principal right ideals and others that are not.

EXAMPLE 5. *Let $G_i = \langle y \rangle \oplus \langle x \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ be the direct sum of two infinite cyclic groups $\langle y \rangle$ and $\langle x \rangle$ with $y^n x^m > e_{G_1}$ if $n > 0$ or $n = 0$ and $m > 0$, and let $G_2 = \langle z \rangle$ also be infinite cyclic. If we choose $N_1 = G_1^+$ (N_2 does not have to be defined) than R as constructed in the above theorem for $H = H_{i=1,2}(G_i, N_i)$ has the following prime ideals:*

$$\hat{x}R \supset (yx^{-n} \mid n = 1,2,\dots) \supset \hat{z}R \supset (0).$$

If we choose $N_1 = G_1$, then

$$\hat{x}R \supset \left(\widehat{yx^{-n}} \mid n = 1,2,\dots\right) \supset \left(\widehat{zy^{-n}} \mid n = 1,2,\dots\right) \supset (0)$$

are the prime ideals. In general, if $i \in I$ has an immediate predecessor in I, if G_i has a smallest positive element $g_{i,\min}$, and if $N_j = G_j^+$ for all $j < i$, then $P_i = \widehat{g_{i,\min}}R$ is principal.

Remark 6. We can extend the definition of a split holoid to also include the holoids one obtains if the pair (G_i, N_i) is replaced by the pair (K_i, K_i) where K_i is a right cancellative holoid; even though such a holoid K_i is embeddable into an ordered group, it is not necessarily isomorphic to the positive cone of its quotient group. The construction given above applies to this additional case.

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